[2] $y'' + \underbrace{\left(\frac{2x^2+5}{6x}\right)}\, y' - \underbrace{\frac{1}{6x^2}}\, y = 0$

BOTH DISCONT @ $x=0$

$$\lim_{x\to 0} x \cdot \frac{2x^2+5}{6x} = \lim_{x\to 0} \frac{2x^2+5}{6} = \frac{5}{6}$$

$$\lim_{x\to 0} x^2 \cdot -\frac{1}{6x^2} = \lim_{x\to 0} -\frac{1}{6} = -\frac{1}{6}$$

BOTH ANALYTIC @ $x=0$

SO X IS A REGULAR SINGULAR POINT

$$\boxed{6x^2 \sum_{n=0}^{\infty} (n+r)(n+r-1)a_n x^{n+r-2} + (2x^3+5x)\sum_{n=0}^{\infty}(n+r)a_n x^{n+r-1} - \sum_{n=0}^{\infty} a_n x^{n+r}}$$ 1

$$= \boxed{\sum_{n=0}^{\infty} 6(n+r)(n+r-1)a_n x^{n+r} + \sum_{n=0}^{\infty} 2(n+r)a_n x^{n+r+2} + \sum_{n=0}^{\infty} 5(n+r)a_n x^{n+r} - \sum_{n=0}^{\infty} a_n x^{n+r}}$$ $\frac{1}{2}$

$$= \sum_{n=2}^{\infty} 2(n+r-2)a_{n-2}x^{n+r}$$

$$= \boxed{\begin{aligned} &6r(r-1)a_0x^r + 6(r+1)ra_1x^{r+1} \\ &+5ra_0x^r \quad + 5(r+1)a_1x^{r+1} \\ &- a_0x^r \quad - a_1x^{r+1} \end{aligned}}$$ $\frac{1}{2}$

$$+ \boxed{\sum_{n=0}^{\infty}\left(\left(6(n+r)(n+r-1)+5(n+r)-1\right)a_n + 2(n+r-2)a_{n-2}\right)x^{n+r}}$$

$\frac{1}{2}$

$= 0$

$6r(r-1)+5r-1=0$

$\boxed{6r^2-r-1=0}$ 1

$(2r-1)(3r+1)=0$

$\boxed{r=\frac{1}{2}, -\frac{1}{3}}$ $\frac{1}{2}$

$$r=\tfrac12:\ \left(6(n+\tfrac12)(n-\tfrac12)+5(n+\tfrac12)-1\right)a_n+2(n-\tfrac32)a_{n-2}=0,$$
$$n\geq 2$$

$$a_n=\frac{-2(n-\frac32)}{6(n+\frac12)(n-\frac12)+5(n+\frac12)-1}\,a_{n-2}$$

$$=\frac{-2n+3}{6n^2\cancel{-\frac32}+5n\cancel{+\frac52}\cancel{-1}}\,a_{n-2}$$

$$=\boxed{\frac{-2n+3}{n(6n+5)}\,a_{n-2}},\quad n\geq 2$$

$\frac12$

LET $a_0=1,\ a_1=0$

$\hookrightarrow a_3=a_5=a_7=\cdots=0$

$$n=2:\ a_2=\frac{-1}{2(17)}a_0=\boxed{-\frac{1}{2\cdot 17}}$$

$$n=4:\ a_4=\frac{-5}{4(29)}a_2=\boxed{+\frac{1\cdot 5}{(2\cdot4)(17\cdot29)}}$$

$$n=6:\ a_6=\frac{-9}{6(41)}a_4=\boxed{-\frac{1\cdot5\cdot9}{(2\cdot4\cdot6)(17\cdot29\cdot41)}}$$

1 ALTOGETHER

$$a_{2n}=(-1)^n\frac{1\cdot5\cdot9\cdots(4n-3)}{(2\cdot4\cdot6\cdots2n)(17\cdot29\cdot41\cdots(12n+5))},\quad n\geq 1$$

$$y_1=\boxed{x^{\frac12}}\left(\boxed{1+\sum_{n=1}^{\infty}(-1)^n\frac{1\cdot5\cdot9\cdots(4n-3)}{2^n n!\,(17\cdot29\cdot41\cdots(12n+5))}x^{2n}}\right)$$

$\frac12$

1

$r=-\frac{1}{3}$: $\left(6\left(n-\frac{1}{3}\right)\left(n-\frac{4}{3}\right)+5\left(n-\frac{1}{3}\right)-1\right)a_n+2\left(n-\frac{7}{3}\right)a_{n-2}=0,$
$n\geq 2$

$$a_n=\frac{-2\left(n-\frac{7}{3}\right)}{6\left(n-\frac{1}{3}\right)\left(n-\frac{4}{3}\right)+5\left(n-\frac{1}{3}\right)-1}a_{n-2}$$

$$=\frac{-2\left(n-\frac{7}{3}\right)}{6n^2-10n+\cancel{\frac{8}{3}}+5n-\cancel{\frac{5}{3}}-\cancel{1}}a_{n-2}$$

$$=\frac{-\frac{2}{3}(3n-7)}{6n^2-5n}a_{n-2}$$

$$=\boxed{\frac{-2(3n-7)}{3n(6n-5)}a_{n-2}},\ n\geq 2$$

$\frac{1}{2}$

LET $a_0=1$, $a_1=0$

$\hookrightarrow a_3=a_5=a_7=\cdots=0$

$n=2$: $a_2=\frac{-\cancel{2}(-1)}{{}_3\cancel{6}(7)}a_0=\boxed{+\frac{1}{3\cdot 7}}$

1 ALTOGETHER

$n=4$: $a_4=\frac{-\cancel{2}(5)}{{}_6\cancel{12}(19)}a_2=\boxed{-\frac{5}{(3\cdot 6)(7\cdot 19)}}$

$n=6$: $a_6=\frac{-\cancel{2}(11)}{{}_9\cancel{18}(31)}a_4=\boxed{+\frac{5\cdot 11}{(3\cdot 6\cdot 9)(7\cdot 19\cdot 31)}}$

$$a_{2n}=(-1)^{n+1}\frac{5\cdot 11\cdot 17\cdots(6n-7)}{(3\cdot 6\cdot 9\cdots 3n)(7\cdot 19\cdot 31\cdots 12n-5)},\ n\geq 2$$

$$y_2=\boxed{x^{-\frac{1}{3}}}\left(\boxed{1+\frac{1}{3\cdot 7}x^2+\sum_{n=2}^{\infty}(-1)^{n+1}\frac{5\cdot 11\cdot 17\cdots(6n-7)}{3^n n!\,(7\cdot 19\cdot 31\cdots(12n-5))}}\right)$$

$\frac{1}{2}$ 1

[3] $$(2x+1)\sum_{n=0}^{\infty}(n+2)(n+1)a_{n+2}x^n + 4\sum_{n=0}^{\infty}(n+1)a_{n+1}x^n - 3x\sum_{n=0}^{\infty}a_n x^n$$ 1

$$= \sum_{n=0}^{\infty}2(n+2)(n+1)a_{n+2}x^{n+1} + \sum_{n=0}^{\infty}(n+2)(n+1)a_{n+2}x^n + \sum_{n=0}^{\infty}4(n+1)a_{n+1}x^n - \sum_{n=0}^{\infty}3a_n x^{n+1}$$ $\frac{1}{2}$

$$= \sum_{n=1}^{\infty}2(n+1)n\,a_{n+1}x^n + \sum_{n=0}^{\infty}(n+2)(n+1)a_{n+2}x^n + \sum_{n=0}^{\infty}4(n+1)a_{n+1}x^n - \sum_{n=1}^{\infty}3a_{n-1}x^n$$ $\frac{1}{2}$

$$= 2\cdot 1\cdot a_2x^0 + 4\cdot 1\cdot a_1x^0 + \sum_{n=1}^{\infty}\left[(n+2)(n+1)a_{n+2} + \left(2(n+1)n + 4(n+1)\right)a_{n+1} - 3a_{n-1}\right]x^n$$ $\frac{1}{2}$

$$= 0$$

$$2a_2 + 4a_1 = 0 \longrightarrow a_2 = -2a_1$$ 1

$$(n+2)(n+1)a_{n+2} + (2n+4)(n+1)a_{n+1} - 3a_{n-1} = 0, \quad n \geq 1$$

$$a_{n+2} = \frac{3a_{n-1} - (2n+4)(n+1)a_{n+1}}{(n+2)(n+1)}$$

$$= \frac{3}{(n+2)(n+1)}a_{n-1} - 2a_{n+1}$$ 1

LET $a_0 = 1, a_1 = 0$

$$a_2 = -2(0) = 0$$

$n=1$: $a_3 = \frac{3}{3\cdot 2} a_0 - 2a_2 = \frac{3}{3\cdot 2}(1) - 2(0) = \frac{1}{2}$

$n=2$: $a_4 = \frac{3}{4\cdot 3} a_1 - 2a_3 = \frac{3}{4\cdot 3}(0) - 2(\frac{1}{2}) = -1$ $1\frac{1}{2}$

$n=3$: $a_5 = \frac{3}{5\cdot 4} a_2 - 2a_4 = \frac{3}{5\cdot 4}(0) - 2(-1) = 2$

$y_1 = \boxed{1 + \frac{1}{2}x^3 - x^4 + 2x^5 + \cdots}$ 1

LET $a_0 = 0, a_1 = 1$

$$a_2 = -2(1) = -2$$

$n=1$: $a_3 = \frac{3}{3\cdot 2} a_0 - 2a_2 = \frac{3}{3\cdot 2}(0) - 2(-2) = 4$

$n=2$: $a_4 = \frac{3}{4\cdot 3} a_1 - 2a_3 = \frac{3}{4\cdot 3}(1) - 2(4) = -\frac{31}{4}$ $1\frac{1}{2}$

$y_2 = \boxed{x - 2x^2 + 4x^3 - \frac{31}{4}x^4 + \cdots}$ 1

$\boxed{y = c_1 y_1 + c_2 y_2}$ $\frac{1}{2}$